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About recurrence time for a semi-Lagrangian discontinuous Galerkin Vlasov solver

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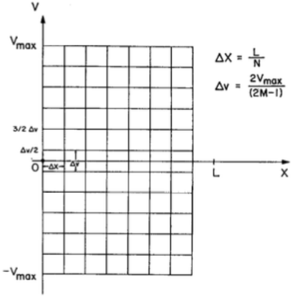
Joint work with Laurent Navoret (IRMA)

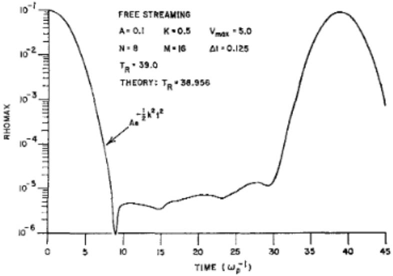
Introduction : results from Cheng-Knorr, 1976

Recurrence time $T_R = \frac{2\pi}{k\Delta v}$ is analyzed for free streaming case...

$$\partial_t f + v\partial_x f = 0,\ \rho = \int f dv,\ f(t=0) = A\cos(kx)f_0(v),\ f_0(v) = \frac{1}{\sqrt{2\pi}}e^{-v^2/2}$$

$$\rho(t,x) = A\Delta v \sum_{j=-M}^{M-1} f_0(v_j)\cos(k(x-(j+1/2)\Delta vt)) \text{ is } T_R \text{ periodic in time}$$





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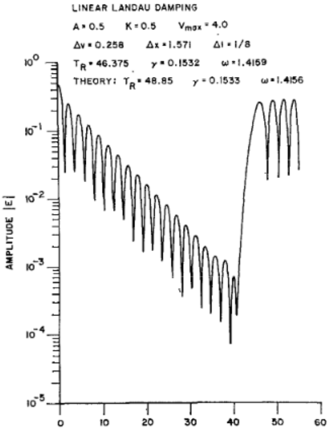
Introduction : results from Cheng-Knorr 1976

...and observed for the simulation of the linear Landau damping

$$\partial_t f + v\partial_x f + E\partial_v f = 0,\ \partial_x E = \rho - 1$$

$$f(t=0) = (1 + A\cos(kx))f_0(v)$$



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Comments and objective

- Numerical artefact for several grid-based solvers
- How to suppress/reduce the phenomenon ?
  - randomness of the grid
  - filtering
  - PIC method
  - finer grid
- Recent works : Einkemmer-Ostermann (2014), Cai-Wang (2017)
- Eliasson (2001, 2010)
- Objective here : study the behavior of the recurrence for a *Semi-Lagrangian Discontinuous Galerkin* (SLDG) method

The SLDG method for Vlasov-Poisson : history

- Rossmanith-Seal $\rightarrow$ Qiu-Shu $\rightarrow$ Crouseilles-M-Vecil (2010-2011)
- already ideas of SLDG in Mangeney-Califano-et-al (2002)
- variant+analysis : Einkemmer-Ostermann (2014)
- adaptive : Besse-Deriaz-Madaule (2016)
- no splitting : Cai-Guo-Qiu (2017)
- other contexts :
  - general approach : Restelli-Bonaventuro-Sacco (2006)
  - Guo-Nair-Qiu (2014)
  - analysis : Bokanowski-Simarmata, Bokanowski-Cheng-Shu (2016)
- some ref. for DG for Vlasov :
  - Heath-Gamba-et-al (2012)
  - Cheng-Christlieb-Zhong, Madaule-Restelli-Sonnendrücker (2014)
  - analysis : Ayuso-Carillo-Shu (2012)

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The SLDG method for Vlasov-Poisson : the scheme

- In $1D$ for free streaming
  - polynomial representation on each cell
  - exact transport (constant velocity)
  - projection on each cell : piecewise polynomial $\rightarrow$ polynomial
  - can be interpreted as advection of Gauss points [1]

$$x_{j,0},\ldots,x_{j,d}\in]x_{\min}+j\Delta x, x_{\min}+(j+1)\Delta x[$$

and written as

$$\begin{pmatrix} f(t+\Delta t, x_{j,0}) \\ \cdots \\ f(t+\Delta t, x_{j,d}) \end{pmatrix} = A(\alpha)\begin{pmatrix} f(t, x_{i,0}) \\ \cdots \\ f(t, x_{i,d}) \end{pmatrix} + B(\alpha)\begin{pmatrix} f(t, x_{i+1,0}) \\ \cdots \\ f(t, x_{i+1,d}) \end{pmatrix}$$

with $x_{j,\cdot}-v\Delta t = x_{i,\cdot}+\alpha\Delta x$ and $A(\alpha), B(\alpha)\in\mathbb{R}^{d+1}\times\mathbb{R}^{d+1}$

- In $1D\times 1D$ phase-space for Vlasov-Poisson
  - polynomial represented by Gauss points on each rectangular cell
  - splitting : succession of
    - $1D$ advections of Gauss points in x direction (each $1D$ advection updates its horizontal line)
    - Poisson
    - $1D$ advections of Gauss points in v direction (each $1D$ advection updates its vertical line)

1. we can use other points similarly

Free streaming for Gauss type mesh

The mesh in velocity is given by

$$v_{j,\ell} = (j + \bar{v}_\ell)\Delta v, \ \ell = 0, \dots, d$$

with $0 < \bar{v}_0 < \cdots < \bar{v}_d < 1$ and we solve

$$\partial_t f + v_{j,\ell}\partial_x f = 0$$

- Framework not only for SLDG : what counts here is the mesh
- Do we have a recurrence time ?
- is it formula $T_R = 2\pi/(k\Delta v)$?
  - OK for $d = 0$ and $\bar{v}_0 = 1/2$.
  - but looking at uniform cases :
    - $d = 1$ and $\bar{v}_0 = 1/4, \ \bar{v}_1 = 3/4, \to T_R = 2\pi/(k\Delta v/2)$
    - $d = 2$ and $\bar{v}_0 = 1/6, \ \bar{v}_1 = 3/6, \ \bar{v}_2 = 5/6, \to T_R = 2\pi/(k\Delta v/3)$
    - $\dots \ \bar{v}_k = (2\ell + 1)/(2d + 2), \ \ell = 0, \dots, d$

  we can hope formula $T_R = 2\pi(d + 1)/(k\Delta v)$
- We will see that formula is between :

$$T_R \simeq 2\pi(\lfloor d/2 \rfloor + 1)/(k\Delta v)$$

Recurrence & quadrature

We get

$$\rho(t,x) = A\Delta v \sum_{j=-M}^{M-1} \sum_{\ell=0}^{d} \omega_\ell f_0(v_{j,\ell}) \cos(k(x-(j+\bar{v}_\ell)\Delta v t))$$

With symmetry of discrete velocity distribution, we obtain

$$\rho(t,x) = A\cos(kx)h(t),$$

with

$$h(t) = \Delta v \sum_{j=-M}^{M-1} \sum_{\ell=0}^{d} \omega_\ell f_0(v_{j,\ell}) \cos(k(j+\bar{v}_\ell)\Delta v t))$$

We look the expression for $t = t_m = \frac{2\pi m}{k\Delta v},\ m \in \mathbb{N}^*$

$$h(t_m) = \sum_{\ell=0}^{d} \omega_\ell \cos(2\pi m \bar{v}_\ell) \Delta v \sum_{j=-M}^{M-1} f_0(v_{j,\ell})$$

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Recurrence & quadrature

As $\Delta v \sum_{j=-M}^{M-1} f_0(v_{j,\ell}) \to \int f_0(v)dv$ exponentially, we have

$$h(t_m) = h_m \int f_0(v)dv + O(\Delta v^p), \;\; h_m = \sum_{\ell=0}^{d} \omega_\ell \cos(2\pi m \bar{v}_\ell)$$

- Energy restituted at $t_m = \frac{2\pi m}{k\Delta v}$, $m \in \mathbb{N}^*$ is $\simeq$ proportional to h_m.
- h_m is a quadrature formula for $\int_0^1 \cos(2\pi m v)dv$
- At continuous level, we have

$$\int_0^1 \cos(2\pi m v)dv = 0 \;\; : \;\; \text{no recurrence}$$

$\Rightarrow$ link between recurrence effect and quadrature rule

- For uniform mesh :
  - $\cos(2\pi m \cdot)$ exactly reproduced by standard quadrature for $m = 0, 1, \ldots, d$:

$$\frac{1}{d+1} \sum_{\ell=0}^{d} \cos(2\pi m \frac{2\ell+1}{2d+2}) = \begin{cases} 0, \text{ if } m = q(d+1)+r, \; r = 1, \ldots, d \\ (-1)^q, \text{ if } m = q(d+1). \end{cases}$$

  - consistent with $T_R = 2\pi(d+1)/(k\Delta v)$.

Trigonometric quadrature

We look for a quadrature rule such that

$$\sum_{\ell=0}^{d} \omega_\ell \cos(2\pi m \bar{v}_\ell) = \int_0^1 \cos(2\pi m v) dv,$$

for all $m \leq n$ with n as high as possible.

- Possible for $n = \lfloor d/2 \rfloor$ and

$$\omega_\ell = \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{j=0,\ j\neq\ell}^{\lfloor d/2 \rfloor} \frac{\cos(v) - \cos(2\pi \bar{v}_j)}{\cos(2\pi \bar{v}_\ell) - \cos(2\pi \bar{v}_j)} dv,\ \ \ell = 0, \ldots, d.$$

- Optimal points are uniform and lead to $n = d$ (like Gauss integration)
- for Gauss points, weights remain positive until $d = 18$.
- recent references : Peherstorfer, 2011 ; Austin, PhD 2016

$\Rightarrow$ Instead of using classical *polynomial* integration quadrature rule, we can use this *trigonometric* integration

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Numerical results for free streaming

$d = 2$ (gauss points)

	classical	new
h_1	0.02245	0
h_2	0.52996	0.51916
h_3	0.73652	0.73047

$d = 3$ (gauss points)

	classical	new
h_1	$-1.0678 \cdot 10^{-3}$	0
h_2	-0.12573	-0.12664
h_3	-0.74097	-0.74155
h_4	-0.33811	-0.33830

$d = 4$ (gauss points)

	classical	new
h_1	$3.0803 \cdot 10^{-5}$	0
h_2	0.016669	0
h_3	0.30402	0.30657
h_4	0.79870	0.80082
h_5	-0.010458	-0.018835

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Numerical results for free streaming

$d = 5$ (gauss points)

	classical	new
h_1	$-5.9704 \cdot 10^{-7}$	0
h_2	$-1.4223 \cdot 10^{-3}$	0
h_3	-0.071038	-0.073117
h_4	-0.50843	-0.50769
h_5	-0.67454	-0.67455
h_6	0.30953	0.30852

$d = 6$ (gauss points)

	classical	new
h_1	$8.3166 \cdot 10^{-9}$	0
h_2	$8.5037 \cdot 10^{-5}$	0
h_3	0.010947	0
h_4	0.17907	0.18741
h_5	0.66873	0.66483
h_6	0.40512	0.40575
h_7	-0.41364	-0.41456

Numerical results for free streaming

- Gauss quadrature (classical) already good
- Gauss quadrature becomes better when d increases
- small improvement for new trigonometric quadrature
- do not use when $d > 18$ (negative weights)
- T_R coherent with result of Madaule-Restelli-Sonnendrücker (2014)

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Gauss-Lobatto points

The analysis can be extended to Gauss-Lobatto points
Example : $d = 2$, the mesh is uniform :

$$x_0 = 0,\ x_1 = 1/2,\ x_2 = 1$$

- Trigonometric interpolation corresponds to trapezoidal formula :

$$\omega_0 = \omega_2 = 1/4, \omega_1 = 1/2$$

- Gauss-Lobatto quadrature is Simpson rule :

$$\omega_0 = \omega_2 = 1/6, \omega_1 = 2/3$$

$d = 2$ (gauss-lobatto points)

	classical	new
h_1	$1/3$	0
h_2	1	1
h_3	$1/3$	0
h_4	1	1

Numerical results for free streaming

$d = 4$ (gauss-lobatto points)

	classical	new
h_1	$1.3174 \cdot 10^{-3}$	0
h_2	0.14855	0
h_3	0.79651	0.93393
h_4	0.25734	0.12776
h_5	-0.10028	-0.11932

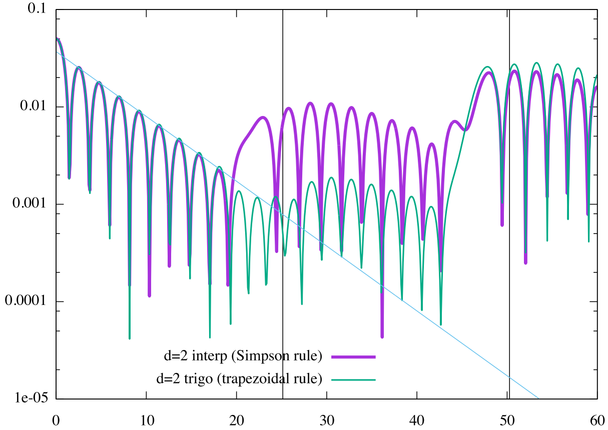
$d = 6$ (gauss-lobatto points)

	classical	new
h_1	$6.9354 \cdot 10^{-7}$	0
h_2	$1.6300 \cdot 10^{-3}$	0
h_3	0.079357	0
h_4	0.542841	0.663454
h_5	0.64533	0.51383
h_6	-0.34435	-0.22468
h_7	0.15274	$2.1373 \cdot 10^{-3}$

positive weights until $d = 7$.

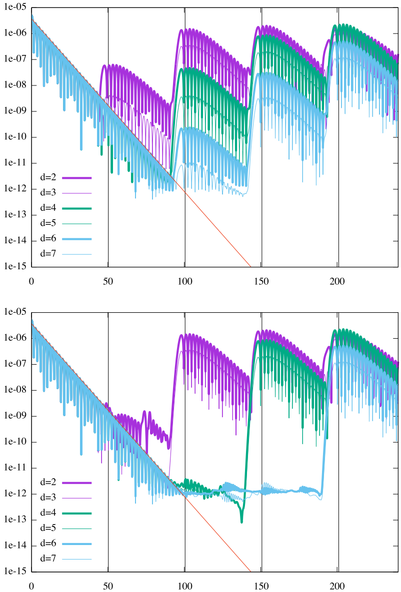
Numerical results for linear Landau damping

$$N = 16,\ v_{\max} = 4,\ A = 10^{-2},\ k = 0.5,\ 2\pi/(k\Delta v) \simeq 25.13$$



Numerical results for linear Landau damping

$N = 64,\ v_{\max} = 8,\ A = 10^{-6},\ k = 0.5,\ 2\pi/(k\Delta v) \simeq 50.26$



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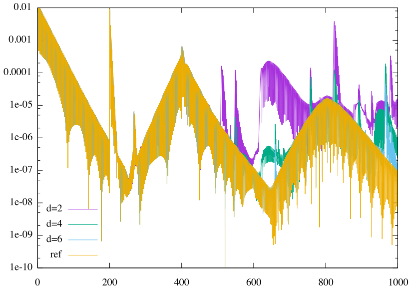
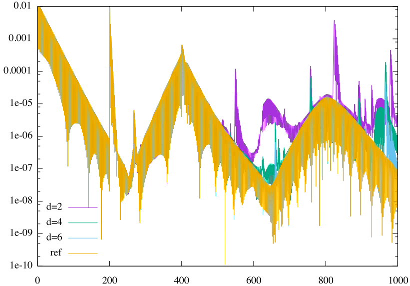
Numerical results for plasma echo

$N = 600,\ v_{\max} = 8,\ A = 10^{-3},\ k_1 = \frac{12\pi}{100}$ perturbation with $k_2 = \frac{24\pi}{100}$ at time $t = 200$ is added. $2\pi/(k_1 \Delta v) = 625$





Conclusion and perspectives

- Conclusion
  - link between recurrence and quadrature
  - introduction of trigonometric quadrature for non uniform velocity discretization
  - analysis for free streaming
  - numerical results for Landau damping
- Perspectives/possible applications
  - velocity semi-discretization method (N. Pham, PhD 2016)
  - curvilinear grid
  - DG, NUFFT

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